Boussinesq equations in a 2-D vertical plane with a mean, imposed stratification

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1 Primitive variable formulation

Formulation of the dimensionless Boussinesq equations for velocity $\boldsymbol{u} = (u, w)$, buoyancy perturbation θ, and pressure p. Lengths are scaled by L_0, velocities by U_0, and the buoyancy is scaled by $g\Delta\rho/\rho_0 L_0$ such that the full, dimensional buoyancy field is expressed as

$$b^* = -\frac{g(\rho^* - \rho_0)}{\rho_0} = \frac{g\Delta\rho}{\rho_0}\left(z + \theta\right), \tag{1}$$

where $\partial\overline{b^*}/\partial z = g\Delta\rho/\rho_0 L_0 \equiv N_0^2$ is the constant, mean, imposed buoyancy gradient. The pressure is scaled with the inertial scaling $P_0 = \rho_0 U_0^2$.

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u, \tag{3}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re}\nabla^2 w + Ri_0\theta, \tag{4}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = \frac{1}{RePr}\nabla^2 \theta - w, \tag{5}$$

where the Reynolds, Prandtl and Richardson numbers are

$$Re = \frac{L_0 U_0}{\nu}, \qquad Pr = \frac{\nu}{\kappa}, \qquad Ri_0 \equiv \left(\frac{1}{Fr_0}\right)^2 = \left(\frac{N_0 L_0}{U_0}\right)^2 \equiv \frac{g\Delta\rho L_0}{\rho_0 U_0^2}. \tag{6}$$

Equations (3)-(5) can be time-stepped in Fourier space:

$$\partial_t \hat{u} = -ik_x\widehat{uu} - ik_z\widehat{wu} - ik_x\widehat{p} - \frac{k_x^2 + k_z^2}{Re}\hat{u}, \tag{7}$$

$$\partial_t \widehat{w} = -ik_x\widehat{uw} - ik_z\widehat{ww} - ik_z\widehat{p} - \frac{k_x^2 + k_z^2}{Re}\widehat{w} + Ri_0\hat{\theta}, \tag{8}$$

$$\partial_t \hat{\theta} = -ik_x\widehat{u\theta} - ik_z\widehat{w\theta} - \frac{k_x^2 + k_z^2}{RePr}\hat{\theta} - \widehat{w}. \tag{9}$$

The pressure in this system acts as a Lagrange multiplier to maintain the incompressibility condition (2). We can update the pressure by choosing it to maintain $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ at each time step. This can be done by splitting the time step; first computing the contribution from all other terms in the momentum equations:

$$\widehat{\boldsymbol{u}_*} = \widehat{\boldsymbol{u}_n} + \Delta t \left(-ik_x \widehat{u_n \boldsymbol{u}_n} - ik_z \widehat{w_n \boldsymbol{u}_n} - \frac{k_x^2 + k_z^2}{Re} \widehat{\boldsymbol{u}_n} + Ri_0 \widehat{\theta_n} \hat{\boldsymbol{z}} \right), \tag{10}$$

and then adding on the pressure gradient afterwards

$$\widehat{\boldsymbol{u}_{n+1}} = \widehat{\boldsymbol{u}_*} - i\boldsymbol{k} \Delta t \widehat{p}. \tag{11}$$

We choose the pressure as the field ensuring $\boldsymbol{\nabla} \cdot \boldsymbol{u}_{n+1} = 0$. Taking the dot product of (11) with $\boldsymbol{k}$, we therefore get

$$\boldsymbol{k} \cdot \widehat{\boldsymbol{u}_{n+1}} = 0 = \boldsymbol{k} \cdot \widehat{\boldsymbol{u}_*} - i|\boldsymbol{k}|^2 \Delta t \widehat{p}, \qquad \Rightarrow \widehat{p} = \frac{-i\boldsymbol{k} \cdot \widehat{\boldsymbol{u}_*}}{\Delta t |\boldsymbol{k}|^2} \tag{12}$$

Essentially, we are solving a Poisson equation for the pressure.

This is presented above for a simple Euler scheme, but can easily be implemented in other numerical methods such as Runge–Kutta.

2 Vorticity-streamfunction formulation

Dealing with the pressure can be avoided by taking the curl of the momentum equation, and considering the flow in terms of its vorticity ζ and streamfunction ψ:

$$\zeta = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}, \qquad \boldsymbol{u} = (u, w) = \left(-\frac{\partial \psi}{\partial z}, \frac{\partial \psi}{\partial x} \right). \tag{13}$$

Solving a Poisson equation is still required in this formulation to update ψ through

$$\nabla^2 \psi = \zeta. \tag{14}$$

The evolution equations in this formulation become

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \zeta}{\partial x} = \frac{1}{Re}\nabla^2 \zeta + Ri_0 \frac{\partial \theta}{\partial x}, \tag{15}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \theta}{\partial x} = \frac{1}{RePr}\nabla^2 \theta - \frac{\partial \psi}{\partial x}. \tag{16}$$

The advantage here is that only three fields (ψ, ζ, θ) are required in the memory rather than the four in the primitive variable formulation (u, w, θ, p). However if one wanted to generalise the stratified problem for 3-D domains, the vorticity-streamfunction formulation could not be applied.

3 Further considerations

3.1 Ultimate convection

Setting $Ri_0 < 0$ enforces a mean unstable buoyancy gradient, driving unbounded convection in the periodic domain. This must be matched by a change in (5) such that the sign of the vertical velocity on the right hand side changes:

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = \frac{1}{RePr}\nabla^2\theta - \mathrm{sgn}(Ri_0)w. \tag{17}$$

3.2 Passive scalars

Setting $Ri_0 = 0$ and removing the vertical velocity on the right of (5) modifies the system such that it simulates an unstratified 2-D plane with a passive scalar. Alternatively, we can keep the stratification and add a second scalar θ_p which does not affect the momentum equation and just satisfies

$$\frac{\partial \theta_p}{\partial t} + u\frac{\partial \theta_p}{\partial x} + w\frac{\partial \theta_p}{\partial z} = \frac{1}{ReSc}\nabla^2\theta_p, \tag{18}$$

where $Sc = \nu/\kappa_p$ is the Schmidt number for the passive scalar.

3.3 Multiple active scalars

We could also add a second scalar that *does* affect the momentum equation. Consider for example a density dependent on temperature and salinity through a linear equation of state:

$$\frac{\rho - \rho_0}{\rho_0} = \left(\beta\overline{S_z} - \alpha\overline{T_z}\right)z + \beta S' - \alpha T'. \tag{19}$$

We impose constant mean gradients $\overline{S_z}$ and $\overline{T_z}$, and solve for the periodic perturbations S' and T'. In this case, we need to define two Richardson numbers

$$Ri_T = \frac{\alpha\overline{T_z}L_0^2}{\rho_0 U_0^2}, \qquad Ri_S = \frac{\beta\overline{S_z}L_0^2}{\rho_0 U_0^2}, \tag{20}$$

the signs of which depend on the signs of the mean gradients. In the primitive variable formulation, the evolution equations would be

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla}p + \frac{1}{Re}\nabla^2\boldsymbol{u} + (Ri_T T' - Ri_S S')\hat{\boldsymbol{z}}, \tag{21}$$

$$\frac{\partial T'}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})T' = \frac{1}{RePr}\nabla^2 T' - \mathrm{sgn}(Ri_T)w, \tag{22}$$

$$\frac{\partial S'}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})S' = \frac{1}{ReSc}\nabla^2 S' - \mathrm{sgn}(Ri_S)w. \tag{23}$$

Here $Pr = \nu/\kappa_T$ and $Sc = \nu/\kappa_S$, and the regime of the system depends on the signs of Ri_T and Ri_S as follows.

- $Ri_T > 0$ and $Ri_S < 0$: doubly stably stratified;
- $Ri_T < 0$ and $Ri_S < 0$: diffusive convection;
- $Ri_T > 0$ and $Ri_S > 0$: salt fingering;
- $Ri_T < 0$ and $Ri_S > 0$: doubly convective.

4 Dimensional formulation

In some cases it may be useful to consider the system in its original, dimensional form. Here we present the equations for a multi-scalar stratified system without nondimensionalization:

$$u_t + (uu)_x + (wu)_z = -p_x/\rho_0 + \nu(u_x x + u_z z), \tag{24}$$
$$w_t + (uw)_x + (ww)_z = -p_z/\rho_0 + \nu(w_x x + w_z z) + g(\alpha T' - \beta S')/\rho_0, \tag{25}$$
$$T'_t + (uT')_x + (wT')_z = \kappa_T(T'_x x + T'_z z) - \overline{T_z} w, \tag{26}$$
$$S'_t + (uS')_x + (wS')_z = \kappa_S(S'_x x + S'_z z) - \overline{S_z} w. \tag{27}$$

Here α is the thermal expansion coefficient, β is the salinity contraction coefficient, and κ_T, κ_S are the molecular diffusivities of heat and salt.